A nonlinear stability theorem for baroclinic quasigeostrophic flow

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Recently, several studies have attempted to establish the nonlinear stability of various planetary flows.1-5 All of these analyses have been based on deriving sufficient conditions for the positive definiteness of the second variation of a relevant constrained Hamiltonian describing the basic flow. However, it is known6-8 that this method fails to establish nonlinear stability for infinite-dimensional dynamical systems. Rigorous nonlinear stability theorems require certain convexity hypotheses on the constrained Hamiltonian.9-14 For flows of relevance to geophysical fluid dynamics, correct nonlinear stability proofs have been given for the stratified Euler equations,12 compressible barotropic flows,13 circular vortex patches,14 and multilayer quasigeostrophic flows.15

The principle purpose of this letter is to provide a rigorous nonlinear stability theorem for continuously stratified quasigeostrophic flow including the effects of topography in a bounded or unbounded horizontal domain. (See note added in proof.) These equations describe the essential dynamics for large-scale, low-frequency atmospheric and oceanic motions.16,17

The nondimensional quasigeostrophic equations for a vertically stratified fluid on the beta plane in the absence of heating or dissipation are

\[ \partial_t + J(p, \varphi) \left[ \Delta p + \bar{\rho}^{-1} (\bar{\rho} S^{-1} p_z)_z + \beta y \right] = 0, \]

with

\[ \left[ \partial_t + J(p, \varphi) \right] \left[ p_z - S(z) h(x, y) \right] = 0 \] on \( z = 0, \)

\[ [\partial_t + J(p, \varphi)] p_z = 0 \] on \( z = 1, \)

where \( p = C_0 \) on the smooth horizontal boundary \( D, \) or \( \nabla p \to 0 \) as \( (x^2 + y^2) \to \infty \) if the fluid is horizontally unbounded. The symbols are standard16-18; however, we point out here that \( h(x, y) \) is the topography, \( p \) is the geostrophic pressure, the two-dimensional geostrophic velocity field \( (u, v) = (-p_y, p_x), \) \( J(\varphi, \varphi) = \partial \varphi / \partial \varphi, \) and \( \partial \) are denoted by subscripts. Note that \( \nabla = (\partial_x, \partial_y) \) and \( \Delta = \nabla^2. \)

The nonlinear stability analysis presented here is based on a Hamiltonian formulation of (1)-(3) given in a previous linear analysis by Blumen,19 modified to include the effects of topography and a smooth arbitrary horizontal boundary. The analysis begins by noting that the functions

\[ F_2 = -\int \int \bar{\rho} \left[ \nabla p \cdot \nabla p + S^{-1} (p_z)^2 \right] dV, \]

\[ F_1 = \int \int \bar{\rho} \Phi(z; q) dV, \]

are conserved by (1)-(3),16 where \( \Phi_1, \Phi_2, \) and \( \Phi_3 \) are smooth functions of their arguments; the quasigeostrophic potential vorticity is given by \( q = \Delta p + \bar{\rho}^{-1} \times (\bar{\rho} S^{-1} p_z)_z + \beta y, \) \( n \) is the outward unit normal on \( D, \)

\( dV = dx \, dy \, dz, \) and \( dA = dx \, dy. \)

The Hamiltonian is given by

\[ H = E_0 + F_1 + F_2 + F_3 + F_4, \]

and an equilibrium solution of (1)-(3) is denoted as \( p'. \) The Hamiltonian can be chosen as (4) since \( F_1, F_2, \) and \( F_3 \) are Casimir functions with respect to the mass-constrained Hamiltonian \( E_0 + F_4 \) and Poisson bracket given in Ref. 20. The derivative of \( H \) evaluation at \( p' \) is given by (after integration by parts)

\[ DH(p') \delta p = \int \int \bar{\rho} \delta q \left[ \Phi'(q') - p' \right] dV \]

\[ + \int \{ \bar{\rho} S^{-1} \delta p_z \left[ p' - \Phi'_z (p'_z) \right] \}_z \ dA \]

\[ + \int \{ \bar{\rho} S^{-1} \delta p_z \] \[ \times \left[ \Phi'_z (p'_z - Sh) - p' \right] \}_z \ dA \]

\[ + \int \delta p \left[ C_0 - \lambda \right] n \delta p \ ds \ dx, \]

where \( \delta p = p - p', \delta q = q - q', \) and \( \left( \left( \right) \right) \) means differentiation with respect to the argument. The stationary solution \( p' \) is a critical point of \( H \) when

\[ \Phi'_z (q') = p', \]

\[ \Phi'_z (p'_z) = p' \] on \( z = 1, \)

\[ \Phi'_z (p'_z - Sh) = p' \] on \( z = 0, \)

and when \( \lambda = C_0. \) Equations (5)-(7) serve to define the functions \( \Phi_1, \Phi_2, \) and \( \Phi_3 \) given \( p'. \)

Nonlinear stability is proved by assuming the convexity hypotheses

\[ 0 < \alpha_1 \Phi'_z < \alpha_2 < \infty, \]

\[ 0 < \mu_1 < -\Phi'_z < \mu_2 < \infty, \]
and examining the conserved functional

$$\hat{H}(\delta p) \equiv H(p' + \delta p) - H(p') - DH(p') \delta p. \tag{11}$$

Here \( \hat{H} \) is conserved since \( H \) is conserved and \( DH(p') \delta p = 0 \) for \( p' \) satisfying (5)–(7). The conditions (8)–(10) are assumed to hold for all arguments and at all times for which smooth solutions exist to (1)–(3). When attention is restricted to a basic state corresponding to a zonal flow [i.e., \( p' = p'(y, z) \)], the convexity assumptions (8)–(10) can be shown to be equivalent to Pedlosky's sufficient linear stability conditions for a zonal flow.21

It follows from (11), exploiting the convexity hypotheses (8)–(10), that

$$2E_\delta + \alpha_1 \int \int [\hat{p}(\delta q)^2] dV$$
$$+ \mu_1 \int \int [\hat{p}S^{-1}(\delta p_z)^2]_{z=1} dA$$
$$+ \gamma_1 \int \int [\hat{p}S^{-1}(\delta p_z)^2]_{z=0} dA \leq 2\hat{H}(\delta p) \leq 2E_\delta$$
$$+ \alpha_2 \int \int [\hat{p}S^{-1}(\delta p_z)^2] dV + \mu_2 \int \int [\hat{p}S^{-1}(\delta p_z)^2]_{z=1} dA$$
$$+ \gamma_2 \int \int [\hat{p}S^{-1}(\delta p_z)^2]_{z=0} dA.$$

Since \( \hat{H}(\delta p) = \hat{H}(\delta p_0) \), where \( \delta p_0 = \delta p(t = 0) \), it follows that

$$2E_\delta + \alpha_1 \int \int [\hat{p}(\delta q)^2] dV$$
$$+ \mu_1 \int \int [\hat{p}S^{-1}(\delta p_z)^2]_{z=1} dA$$
$$+ \gamma_1 \int \int [\hat{p}S^{-1}(\delta p_z)^2]_{z=0} dA$$
$$< 2E_\delta + \alpha_2 \int \int [\hat{p}S^{-1}(\delta q_0)^2] dV$$
$$+ \mu_2 \int \int [\hat{p}S^{-1}(\delta p_0_z)^2]_{z=1} dA$$
$$+ \gamma_2 \int \int [\hat{p}S^{-1}(\delta p_0_z)^2]_{z=0} dA,$$  \( \tag{12} \)

which establishes nonlinear stability. The \textit{a priori} estimate (12) implies Lyapunov stability of smooth solutions to (1)–(3) and is explicitly independent of the topography \( h(x,y) \).22

The existence of classical solutions of (1)–(3) in a horizontally periodic domain has been proved only up to a finite time, which is inversely proportional to the norms of \( p_{0z} \) on \( z = 0 \) and \( z = 1 \), and \( q_0 \) (Ref. 23). Stability can also be proved when \( \Phi_\gamma > 0 \) and \( \Phi_\gamma > 0 \) by considering \( -\hat{H}(\delta p) \) and requiring sufficiently large min(\( -\Phi_\gamma \)) and min(\( \Phi_\gamma \)). The stability theorem can be generalized to include "islands" (i.e., non-simply-connected domains) by introducing other circulation functions similar to \( F_e \). Since the quasigeostrophic equations are zonally (i.e., in the \( x \) direction) Galilean invariant, these results also apply to (zonally) steadily translating fluid motions. Therefore the stability theorem presented here is of importance for the nonlinear stability of solitary planetary waves24 (provided \( \Phi_\gamma \), \( \Phi_\gamma \), and \( \Phi_\gamma \) are sufficiently smooth). On this latter application, a more detailed analysis will be published elsewhere.

\textit{Note added in proof:} The author has become aware of a similar stability analysis by M. E. McIntyre and T. G. Shepherd.25

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18. We assume a stably stratified fluid so \( S(x) > 0 \). In the ocean the stratification parameter \( S(x) \) is proportional to \( -\beta \), \( \gamma \). One can set \( \beta = 1 \) in (11). In the atmosphere \( S(x) = \beta \), \( \gamma \), where \( \beta \) is the ambient vertically stratified potential temperature.
21. In the absence of topography (i.e., \( h = 0 \)), this was demonstrated by Blumen. For the more general linear stability result, see Pedlosky (Chap. 7).20 Andrews [Geophys. Astrophys. Fluid Dyn. 28, 243 (1984)] has shown that the only steady quasigeostrophic flows satisfying zonally symmetric boundary conditions for which (8)–(10) hold are zonally symmetric flows.
22. While \( h(x,y) \) does not explicitly enter (12), it implicitly affects the \textit{a priori} bound since \( \gamma_1 \) and \( \gamma_2 \) of (10) depend on \( h(x,y) \).
24. M. E. McIntyre (private communication).